§5.2 The Characteristic Equation

Recall that if A is an $n \times n$ matrix and x is a vector in $\mathbb{R}^n$, then x is an eigenvector of A if

$$Ax = \lambda x$$

for some scalar λ. Here λ is an eigenvalue of A and x is an eigenvector associated to λ.

Last time we saw how to find eigenvectors associated to a given eigenvalue λ by examining the null space of $A - \lambda I$. But how do we find eigenvalues in the first place?

If A is an $n \times n$ matrix, λ is an eigenvalue of A if there's a nonzero vector x with

$$Ax = \lambda x$$

in other words $(A - \lambda I)x = 0$ has a nontrivial solution. Notice $A - \lambda I$ is a square matrix so by the invertible matrix theorem this is equivalent to saying

$$A - \lambda I \text{ is } \underline{\text{not}} \text{ invertible.}$$

Better yet, from chapter 3 this is equivalent to

$$\det(A-\lambda I)=0.$$

Theorem

If A is an $n \times n$ matrix, then λ is an eigenvalue if and only if $\det(A-\lambda I)=0$

Example

Let $A=\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. Last time we showed 3 and -7 were eigenvalues. Let's verify ~~ag~~ this again

$$\det(A-\lambda I)=\det\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$=\det\left(\begin{bmatrix} (2-\lambda) & 3 \\ 3 & (-6-\lambda) \end{bmatrix}\right)$$

$$=(2-\lambda)(-6-\lambda)-(3)(3)$$

$$=-12+4\lambda+\lambda^2-9$$

$$=\lambda^2+4\lambda-21$$

$$=(\lambda-3)(\lambda+7)$$

Thus indeed, $\det(A-\lambda I)=0$ when $\lambda=3$ or $\lambda=-7$, so these are eigenvalues. Moreover these are the only eigenvalues of A.

Defn: Notice that for $n \times n$ matrix A, $\det(A-\lambda I)$ is a polynomial expression of degree n in terms of λ. We ~~calls~~ call this polynomial $p(\lambda) = \det(A-\lambda I)$ the characteristic polynomial (or equation) of A.

Notice that the roots of this polynomial are exactly the ~~char~~ eigenvalues of A.

Example

Let $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find the characteristic polynomial of A and the eigenvalues of A.

Soln'

$$\det(A-\lambda I) = \det \begin{bmatrix} (5-\lambda) & -2 & 6 & -1 \\ 0 & (3-\lambda) & -8 & 0 \\ 0 & 0 & (5-\lambda) & 4 \\ 0 & 0 & 0 & (0-\lambda) \end{bmatrix}$$

$$= (5-\lambda)^2(3-\lambda)(-\lambda)$$

$$= \boxed{\lambda(\lambda-3)(\lambda-5)^2}$$

(Just pulling out −'s. Not necessary!)

Thus the eigenvalues of A are $0, 3$, and 5. Which agrees with the last theorem from last time!

Remarks

- Recall 0 is an eigenvalue if and only if A is not invertible. In the last example, we see this since A has a row of all zeroes.

- In the previous example, the factor $(\lambda-5)$ appears twice in the characteristic polynomial. We say the <u>(algebraic) multiplicity</u> of $\lambda=5$ is 2 in this case.

  In general, the multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Observations

- Since $\det(A-\lambda I)$ is a polynomial of degree n (for A $n\times n$), there are at most n eigenvalues for A (counting multiplicity).

- There are matrices with no eigenvalues! For example, let

$$A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2+1$$

which has no <u>real</u> roots. However, it does have complex roots! More on this in §5.5

<u>Defn</u>

If A and B are $n \times n$ matrices, we say A is <u>similar to B</u> if there exists an invertible $n \times n$ matrix P with

$$A = PBP^{-1}$$

Notice that if A is similar to B and $A = PBP^{-1}$, then $B = P^{-1}AP = (P^{-1})A(P^{-1})^{-1}$ so B is also similar to A. Thus we usually just say A and B are <u>similar</u> to each other.

As it turns out, similar matrices have the same eigenvalues!

Theorem

Let A and B be similar $n \times n$ matrices. Then A and B have the same characteristic polynomial and hence the same eigenvalues.

Proof

Write $A = PBP^{-1}$ and we show $\det(A - \lambda I) = \det(B - \lambda I)$

Notice

$$A - \lambda I = PBP^{-1} - \lambda I = PBP^{-1} - \lambda PP^{-1} = P(B - \lambda I)P^{-1}$$

Thus

$$\begin{aligned} \det(A - \lambda I) &= \det\left(P(B - \lambda I)P^{-1}\right) \\ &= \cancel{\det(P)} \cdot \det(B - \lambda I) \cdot \cancel{\det(P^{-1})} \\ &= \det(B - \lambda I) \end{aligned}$$

Recall $\det(P^{-1}) = \dfrac{1}{\det P}$